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On the Minimal Weight of Binary Group Codes

by

L. Calabi and E. Myrvaagnes

Air Force Cambridge Research Laboratories
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Bedford, Massachusetts

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Abstract

Let $w(n,k)$ be the largest integer such that there exists a binary group code (n,k) all of whose non-zero elements have weight equal to or larger than $w(n,k)$.

In this report values of $w(n,k)$ are given for $0 < k \leq 6$ and $k \leq n \leq 100$, as well as for $0 < k \leq n \leq 24$. Further, new upper and lower bounds are obtained which are easy to compute and, in certain regions, better than other known bounds.

Table of Contents

	Introduction	iii
1.	Functional relations satisfied by w	1
2.	Study in the region $0 < k \le n \le 2^k$	9
3.	Study in the region $k \le n \le 2^{k-1}$	17
4.	Some scattered values of w	24
5.	The function w for $k \le 4$	26
6.	The region in which $w(n, k) \le 4$	29
7.	A small table of values	31

Introduction

We study here the function w that associates with each pair of integers n, k with $0 < k < n$, the largest integer $w(n,k)$ such that there exists a binary group code $A(n,k)$ each of whose non-zero elements has a weight not smaller than $w(n,k)$.

Great efforts have been spent by many scientists in the investigation of w and of equivalent or related notions. Sophisticated and powerful tools have been used to derive bounds or some isolated values. Still, rather little is known.

We have approached the topic afresh, starting with the naive ambition to describe w by formulas or, at least, by extensive tables. Clearly we have not succeeded. But using only very elementary mathematics and hand computation (together with some known results), we have obtained, in particular, new upper and lower bounds, and new values.

The numerical results are given in two tables in the last section: one covers the range $k \leq 6$, $n \leq 100$, and the other the range $0 < k \leq n \leq 24$.

The better bounds, given in section 3, are restricted to the range $0 < k \leq n \leq 2^{k-1}$. They are very easy to compute and, in some regions, are better than any other bound known to us.

We hoped eventually to characterize w by establishing functional equations satisfied by it. We failed, perhaps because the many properties listed turned out to be consequences of only 8 of them. These are satisfied by a class of functions, not only by w. To obtain bounds, then, we have only to find the smallest function and the largest one within the class.

A paper by J. H. Griesmer (A Bound for Error-Correcting Codes, I.B.M.J.R.D., 4, pp.532-542, Nov. 1960) has recently been brought to our attention. We plan to study in a later report the relations between his work and ours.

1. Functional relations satisfied by w.

For any integral valued function f defined for pairs (n, k) of integers satisfying $0 < k \le n$, consider the following functional relations:

1. $f(n,n) = 1$
2. $f(n,k) \le f(n+1, k)$
3. $f(n,k) \le f(n-1, k-1)$
4. $f(n,k) + f(m,k) \le f(n+m, k) \le f(n,k) + m$
5. $f(n,k) = 2h \qquad f(n-1, k) = 2h-1$.

We will now show that w satisfies all five relations and thus all consequences of them, in particular those that we will derive below.

Proposition 1 The function w satisfies 1. to 5.
The proofs of 1. and 2. are trivial. To prove 3., delete the first row and the first column of the generating matrix in "echelon form" of an (n, k) code: what remains is the generating matrix of an $(n-1, k-1)$ code, with not-smaller minimal weight.
The first part of 4. can be obtained by juxtaposing the generating matrices of an (n,k) and an (m,k) code; and the second part is obvious.
Finally 5., also well known, is proven by adding an "overall parity check" to the elements of a code $(n-1, k)$ with $w(A) = w(n-1, k)$.
Because of Proposition 1, part of the study of w can be considered as the study of the relations 1. - 5.

Proposition 2 Relations 2. and 3. imply

6. $f(n,k) \le f(n, k-1)$
7. $f(n-1, k+1) \le f(n,k)$.

In fact $f(n,k) \le f(n+1, k) \le f(n, k-1)$ yields 6.; which, with 2., gives $f(n-1, k+1) \le f(n-1, k) \le f(n,k)$.

Proposition 3 If $0 < h \le k$ and $h \le n-k$, then relations 3. and 4. imply

8. $f(n,k) \le h + f(n-k, h)$.

Indeed, $f(n,k) = f(n-k+l+(k-l),\ l+(k-l))$
$\leq f(n-k+l,\ l)$ by 3.,
$\leq f(n-k,\ l)+l$ by 4.

<u>Proposition 4</u> Relations 1. and 4. imply

9. $f(n,1) = n$.

If $n=1$, this is 1.: we can thus use induction on n. By 4.:

$$f(n-1,1)+f(1,1) \leq f(n,1) \leq f(n-1,1)+1.$$

Thus

$$f(n,1) = f(n-1,1)+1 = n-1+1 = n.$$

<u>Proposition 5</u> Relations 1., 2., and 4 imply that, given two positive integers k and a, there exists n such that $f(n,k)=a$.
If $a=1$, take $n=k$: by 1., $f(k,k)=1$. We can thus assume the proposition true for all pairs k,b with $b<a$, and use induction. There exists then an integer m such that $f(m,k)=a-1$: let, moreover, m be the largest integer with this property. Then, by 4., $f(m+1,k) \leq f(m,k)+1=a$, but also $f(m+1,k) > f(m,k) = a-1$.

<u>Corollary</u> Relations 1., 2., and 4. imply that, given a positive integer a, there are infinitely many pairs n,k such that $f(n,k)=a$.

<u>Proposition 6</u> Relations 2., 4. and 5. imply

10. $f(n,k) = 2k-2$ if $f(n+1,k) = 2k-1$.

In fact, by 5., $f(n,k)$ has to be even. But 2. and 4. yield

$$f(n+1,k) \geq f(n,k) \geq f(n+1,k)-1 = 2k-2.$$

<u>Corollary</u> If $f(n,k)=2k-1$, relations 2., 4. and 5. imply for neighboring points the values given in the table below:

	$k-1$	k	$k+1$
$n-1$	$\geq 2k-1$	$2k-2$	$\leq 2k-2$
n	$\geq 2k$	$2k-1$	$\leq 2k-2$
$n+1$	$\geq 2k$	$2k$	$\leq 2k-1$

The k-column is given by 5. and Prop. 6; the other two columns by properties 2., 3., and 6.

Lemma 1 Let f, g verify 1., 2., 4. and 5. as well as: $f(n,k) = g(n,k)$ for all n, k at which f has an odd value. Then $f = g$.
If $g(n,k) = 2h-1$, for some n', k we have also $f(n',k) = 2h-1$ by Prop. 5. Thus by assumption $f(n',k) = g(n',k) = g(n,k)$: but then 5. and Prop. 6 imply $n = n'$. Thus f and g agree whenever one of them has an odd value; 2. and Prop. 5 show then $f(n,k) = g(n,k)$ for all pairs n, k.

Lemma 2 Let f, g verify 1., 2., 4. and 5. as well as: $f(n,k) > g(n,k)$ for all n, k at which f has an odd value. Then $f \geq g$.
In fact if $g(n,k) = 2h-1$, for some n', k we have $f(n',k) = 2h-1$. Thus $g(n,k) = f(n',k) > g(n',k)$ and $n > n'$: consequently $f(n,k) \geq f(n',k) + 1 > g(n,k)$. Thus $f > g$ whenever one of them is odd. Let $g(n,k)$ and $f(n,k)$ be both even: there is then $h > 0$ such that

$$g(n+h,k) = g(n,k), \quad f(n+h,k) = f(n,k)+1$$

or

$$g(n+h,k) = g(n,k)+1, \quad f(n+h,k) = f(n,k).$$

In the first case we have

$$f(n,k)+1 = f(n+h,k) > g(n+h,k) = g(n,k);$$

that is, $f(n,k) \geq g(n,k)$.

In the second case

$$f(n,k) = f(n+h,k) > g(n+h,k) > g(n,k),$$

hence the lemma.

A very similar proof yields also:

Lemma 3 Let f, g verify 1., 2., 4. and 5. as well as: $f(n,k) > g(n,k)$ for all n, k at which g has an odd value. Then $f \geq g$.

Combining the last three results we obtain:

Proposition 7 Let f, g verify 1., 2., 4. and 5. as well as: $f(n,k) \geq g(n,k)$

for all n, k at which f [or g] has an odd value. Then $f \geq g$.

Proposition 8 Let $f(n,k)$ be odd. Then 4. and 5. imply equivalence between:

a) $f(n,k) + f(m,k) = f(n+m,k)$

b) $f(n+1,k) + f(m,k) = f(n+m+1,k)$.

Assume a): then b) follows from the inequalities

$$f(n+m,k)+1 = f(n,k) + f(m,k)+1 = f(n+1,k) + f(m,k) \leq$$
$$\leq f(n+m+1,k) \leq f(n+m,k)+1.$$

Conversely, if b) Holds:

$$f(n+m+1,k)-1 = f(n+1,k) + f(m,k)-1 = f(n,k) + f(m,k) \leq$$
$$\leq f(n+m,k) \leq f(n+m+1,k)-1.$$

As a consequence:

Corollary Let $f(n,k)$ and $f(m,k)$ be odd. Then 4. and 5. imply equivalence between:

a) $f(n,k) + f(m,k) = f(n+m,k)$

b) $f(n+1,k) + f(m,k) = f(n,k) + f(m+1,k) = f(n+m+1,k)$

c) $f(n+1,k) + f(m+1,k) = f(n+m+2,k)$.

Set $f_1(n,k) = n-k+1$ for $0 < k \leq n$. Then:

Proposition 9 The function f_1 satisfies 1. to 5. Moreover any function f satisfying 1., 3. and 4. satisfies also

11. $f(n,k) \leq f_1(n,k)$.

For $f = \mu$, this inequality has been proven also, for instance, in [1] and [2]. That f_1 satisfies 1. to 5. is obvious. To establish 11., use 3. to obtain

$$f(n,k) \leq f(n-k+1, 1);$$

Prop. 4 completes the proof.

For $0 < k \leq n$, set now

$$f_0(n,k) = \begin{cases} 2r-1 & \text{if} \quad n = rk + r - 1 \\ 2r & \text{if} \quad rk + r - 1 < n < (r+1)k + r. \end{cases}$$

Equivalently

$$f_0(n,k) = \begin{cases} 2\left[\frac{n+1}{k+1}\right] - 1 & \text{if} \quad n+1 = r(k+1) \\ 2\left[\frac{n+1}{k+1}\right] & \text{if} \quad n+1 = r(k+1) + q, \ 1 \leq q \leq k; \end{cases}$$

or

$$f_0(n,k) = \begin{cases} 2\left[\frac{n-k}{k+1}\right] + 1 & \text{if} \quad n+1 = r(k+1) \\ 2\left[\frac{n-k}{k+1}\right] + 2 & \text{if} \quad n+1 = r(k+1) + q, \ 1 \leq q \leq k. \end{cases}$$

Proposition 10 The function f_0 satisfies 1. to 5. Moreover any function f satisfying 1., 2., 4. and 5. satisfies also

12. $f_0(n,k) \leq f(n,k)$.

Property 3. is the only one whose proof is not immediate. Assume first $n-1 = r(k-1) + r - 1$. Then $f_0(n-1, k-1) = 2r-1$ and $n = rk$. There exists then a smallest integer $a > 0$ such that $n = (r-a)k + (r-a-1) + b$, for some $b \geq 0$. Then $f_0(n,k) = 2(r-a) - 1$ if $b = 0$, $= 2(r-a)$ if $b > 0$. In either case, 4. holds.

Assume now $r(k-1) + r - 1 < n - 1 < (r+1)(k-1) + r + 1$. This implies $f_0(n-1, k-1) = 2r$ and $rk < n < (r+1)k$. Again there is a smallest integer $a > 0$ such that

$$(r-a)k + r - a - 1 < n < (r-a+1)k + r - a$$

and the inequality 4. follows.

In order to establish 12. we will use Prop. 7. Let us then prove $f(n,k) \geq f_0(n,k)$ when $f_0(n,k)$ is odd, that is when $n+1 = r(k+1)$.

If $r=1$, $n=k$ and $f(k,k)=f_0(k,k)$. Assume thus

$$f((r-1)(k+1)-1,k)\geq f_0((r-1)(k+1)-1,k)=2r-3.$$

Since the right hand member is odd, we have also

$$f((r-1)(k+1),k)\geq f_0((r-1)(k+1),k)=2r-2.$$

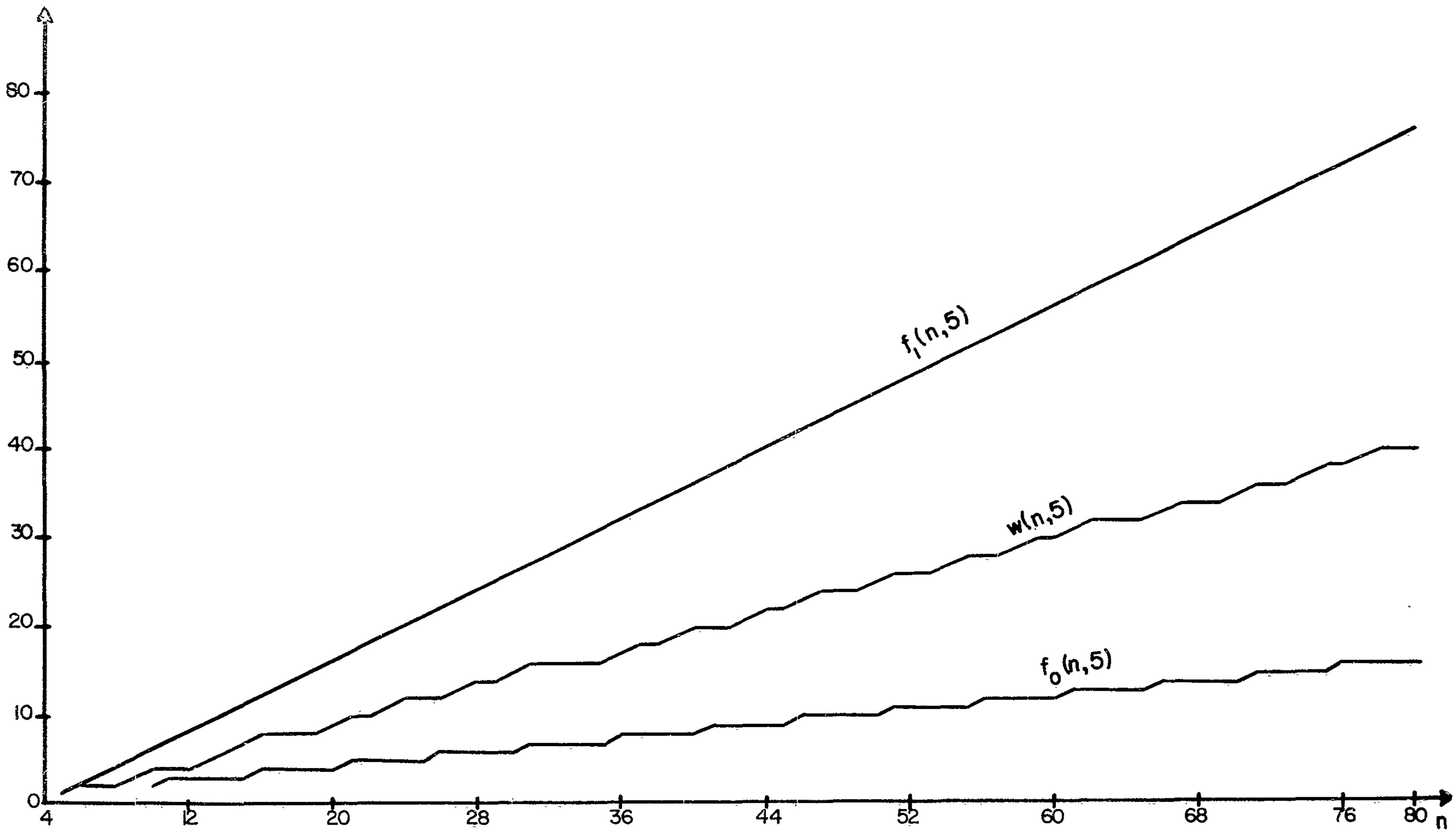
Thus

$$\begin{aligned}f(r(k+1)-1,k)&=f((r-1)(k+1)+k,k)\geq f((r-1)(k+1),k)+f(k,k)\\&=f((r-1)(k+1),k)+1\geq f_0((r-1)(k+1),k)+1=2r-1\\&=f_0(r(k+1)-1,k).\end{aligned}$$

The assumptions of Prop. 7 are hence established and 12. follows.

The inequalities 11. and 12. give easy geometric bounds for the functions with properties 1. to 5. For fixed k, the graph of f_1 is a straight line with slope $1=\tan 45^\circ$; but the graph of f_0 can be interpolated by a straight line $2\frac{n+1}{k+1}-1$, of slope $\frac{2}{k+1}$. Thus, for $k=1$ the two lines coincide; for $k>1$, their "difference" increases with k. If we keep n fixed, f_1 is again a straight line, now with slope -1. But f_0 resembles an hyperbola of the type $\frac{c}{k+1}-1$, which grows closer to its asymptotes (and hence farther away from f_1) the larger $c=2(n+1)$ becomes.

In the following two graphs f_0, f_1 and w have been plotted, once for $k=5$, and next for $n=17$.



80
70
60
50
40
30
20
10
0
4
12
20
28
36
44
52
60
68
76
80
n
$f_1(n,5)$
$w(n,5)$
$f_0(n,5)$

Fig. 1

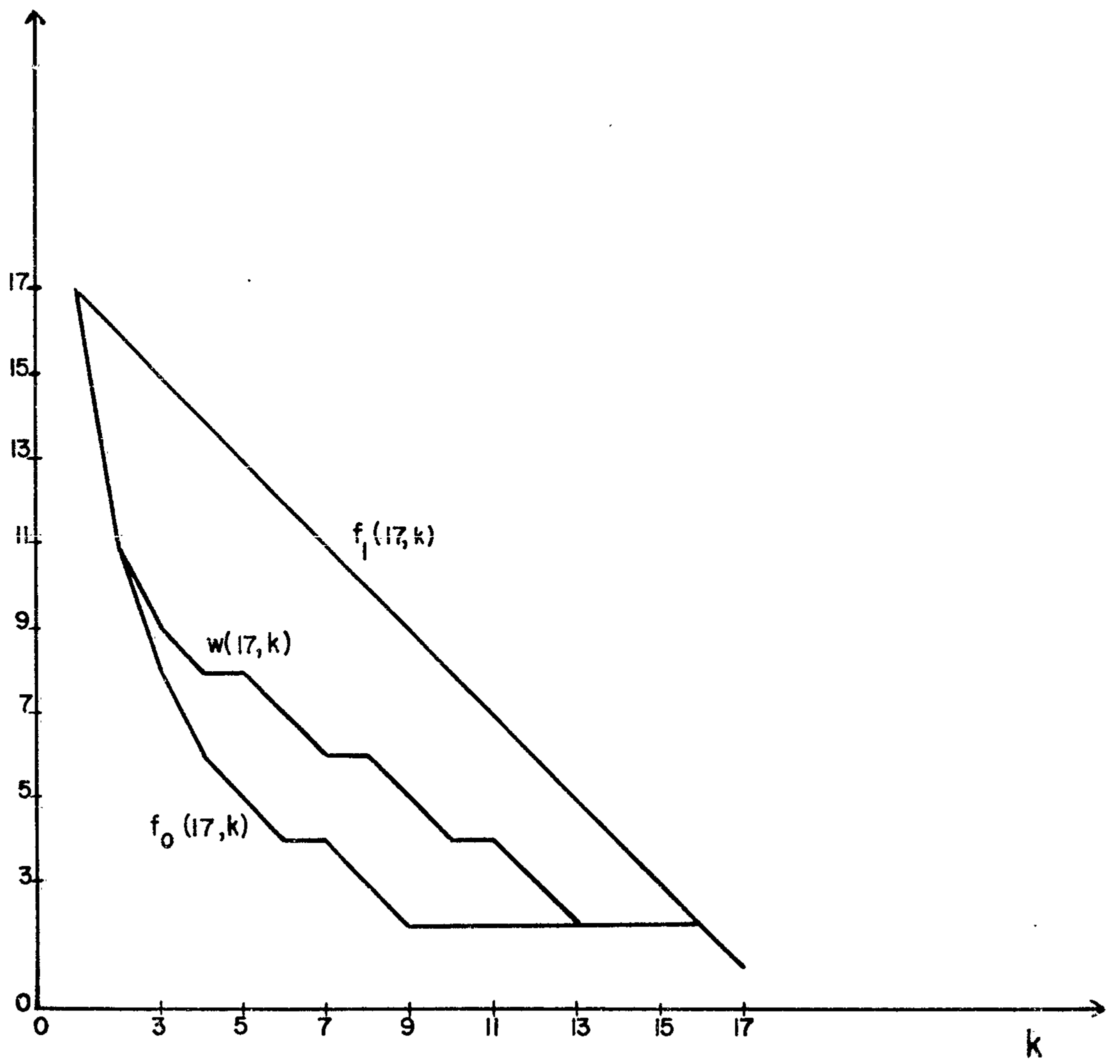


Fig. 2

2. Study in the region $0 < k \leq n \leq 2^k$.

It is well known that w satisfies:

$$13. \quad f(h(2^k - 1), k) = h\, 2^{k-1}, \text{ for } h = 1, 2, \ldots .$$

This is not a consequence of 1. to 5., as, for instance, f_1 shows. An easy proof yields:

Proposition 11 Relations 4. and 13. imply

$$14. \quad f(2^k, k) \leq 2^{k-1} + 1 \text{ and } f(2^h, k) \leq 2^{h-1} \text{ for } h < k \leq 2^h.$$

The first part of the proposition follows directly from 13. and the right-hand part of 4. The second part follows from the first and from the left-hand part of 4:

Letting $r = k - h$, we have

$$2^r f(2^h, k) \leq f(2^r 2^h, k)$$
$$= f(2^k, k) \leq 2^{k-1} + 1.$$

It is also known that the following holds for w:

$$15. \quad f(2^{k-1}, k) = 2^{k-2}$$

Corresponding to Prop. 11, and with a similar proof, we have now:

Proposition 12 Relations 4. and 15. imply

$$16. \quad f(2^k, k) \geq 2^{k-1} \text{ and } f(2^h, k) \leq 2^{h-1} \text{ for } h < k \leq 2^h.$$

For $0 < k \leq n \leq 2^k$ define a function g_0 as follows

$$g_0(n, k) = \max\left(f_0(n, k),\ n - 2^{k-2}\right) \text{ if } n \leq 2^{k-1}$$
$$g_0(n, k) = \max\left(2^{k-2},\ n+1-2^{k-1}\right) \text{ if } 2^{k-1} < n < 2^{k-1} + k$$
$$g_0(n, k) = \max\left(2^{k-2} + f_0(n - 2^{k-1}, k),\ n+1-2^{k-1}\right) \text{ if } 2^{k-1} + k \leq n < 2^k$$
$$g_0(2^k, k) = 2^{k-1}.$$

More explicit, but apparently more cumbersome, formulas can be easily derived. It should not be hard also to prove that g_0, where defined, satisfies 1. to 16.: in fact 3. seems the only non obvious relation.

We are, however, interested only in the following result:

Proposition 13. Let f satisfy 1. to 16.: then

$$17. \quad g_0(n,k) \leq f(n,k) \quad \text{if} \quad 0 < k \leq n \leq 2^k.$$

As long as $g_0 = f_0$; 17. is relation 12., and thus holds. In the next interval, when $g_0(n,k) = n - 2^{k-2}$, we have also

$$g_0(n,k) = 2^{k-2} - r \quad \text{if} \quad n = 2^{k-1} - r.$$

Our result then follows from 15. and 4. Relation 2. implies then 17. as long as $g_0(n,k) = 2^{k-2}$; and 2. with 4. imply 17. when $g_0(n,k) = 2^{k-2} + f_0(n - 2^{k-2}, k)$. Finally 13. and 4. prove our result when $g_0(n,k) = n + 1 - 2^{k-1}$ because this is equivalent to

$$g_0(n,k) = 2^{k-1} - r \quad \text{if} \quad n = 2^k - 1 - r.$$

Finally, 16. shows $g_0(2^k, k) \leq f(2^k, k)$.

It will be shown later that $g_0(n,k) = w(n,k)$ for $k \leq 3$. Some experimental evidence seems to indicate that, if $k > 3$, $\max(w(n,k) - g_0(n,k))$ is probably reached around $n = 2^{k-1} + 2^{k-2}$. Notice that $w(2^{k-1} + 2^{k-2}, k) = 2^{k-2} + 2^{k-3}$ while $g_0(2^{k-1} + 2^{k-2}, k) = f_0(2^{k-1} + 2^{k-2}, k)$ giving a difference approximated by

$$2^{k-2} + 2^{k-3} - 2\,\frac{2^{k-1} + 2^{k-2} + 1}{k+1}$$

or

$$\left(3 - \frac{10}{k+1}\right) 2^{k-3}$$

which tends to $3 \cdot 2^{k-3}$ when k increases.

In parallel to g_0 and Prop. 13, and with similar proof, we obtain an upper bound as follows.

For $0 < k \leq n \leq 2^k$, let

$$g_1(n,k) = \min\left(f_1(n,k), 2^{k-2}\right) \quad \text{for} \quad k \leq n \leq 2^{k-1}$$

$$g_1(n,k) = \min\left(n - 2^{k-2}, 2^{k-1}\right) \quad \text{for} \quad 2^{k-1} < n < 2^k$$

$$g_1(2^k, k) = 2^{k-1} + 1.$$

We have then:

Proposition 14 Let f satisfy 1. to 16.: then

$$18. \quad f(n,k) \leq g_1(n,k) \quad \text{if} \quad 0 < k \leq n \leq 2^k.$$

Again, as for g_0, $\max(g_1(n,k) - w(n,k))$ seems to be reached at $n = 2^{k-1} + 2^{k-2}$, where the difference is

$$2^{k-1} - 2^{k-2} - 2^{k-3} = 2^{k-3}.$$

The considerable improvement from the bounds f_0, f_1 to g_0, g_1 has been obtained by imposing the "boundary values" 13. and 15. It seems natural that further boundary values would yield better and better bounds. Since w is known for $n = 2^k - 2^r$, $r = 2, 3, \ldots, k-1$ we could probably obtain fair approximations to w for $2^{k-1} \leq n \leq 2^k$. But the region $k \leq n \leq 2^{k-1}$ is of much greater practical interest: and, in it, practically nothing is known.

Other, better bounds, could also be obtained if other functional relations satisfied by w were known. A step in this direction is taken in the next section. We terminate this section by plotting g_0, g_1 and, when known, w for $k=6$ and for $n=20$. In fig. 5, 6 and 7, g_0 is compared with the Varsharmov - Gilbert lower bound [3].

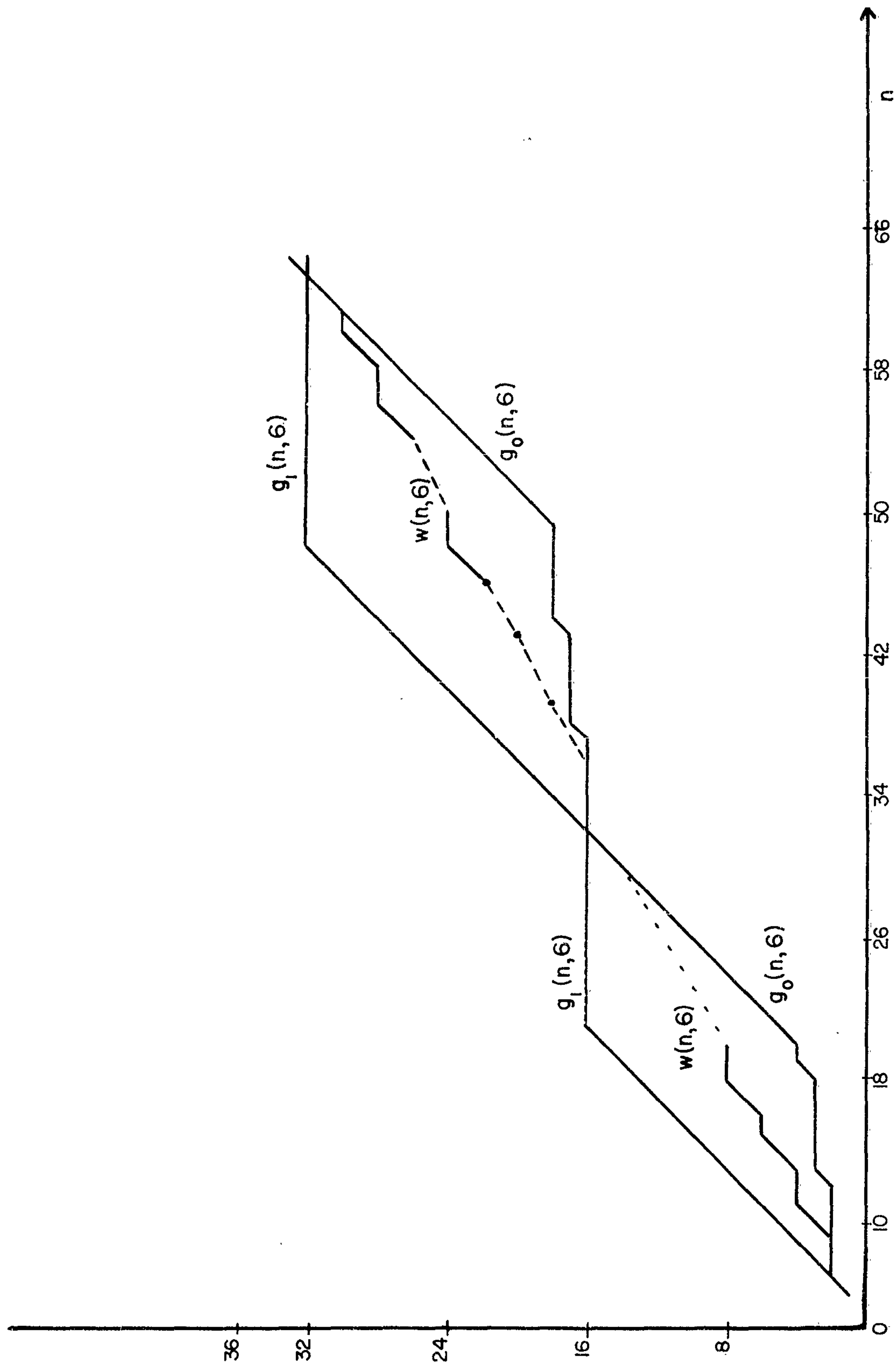


Fig. 3

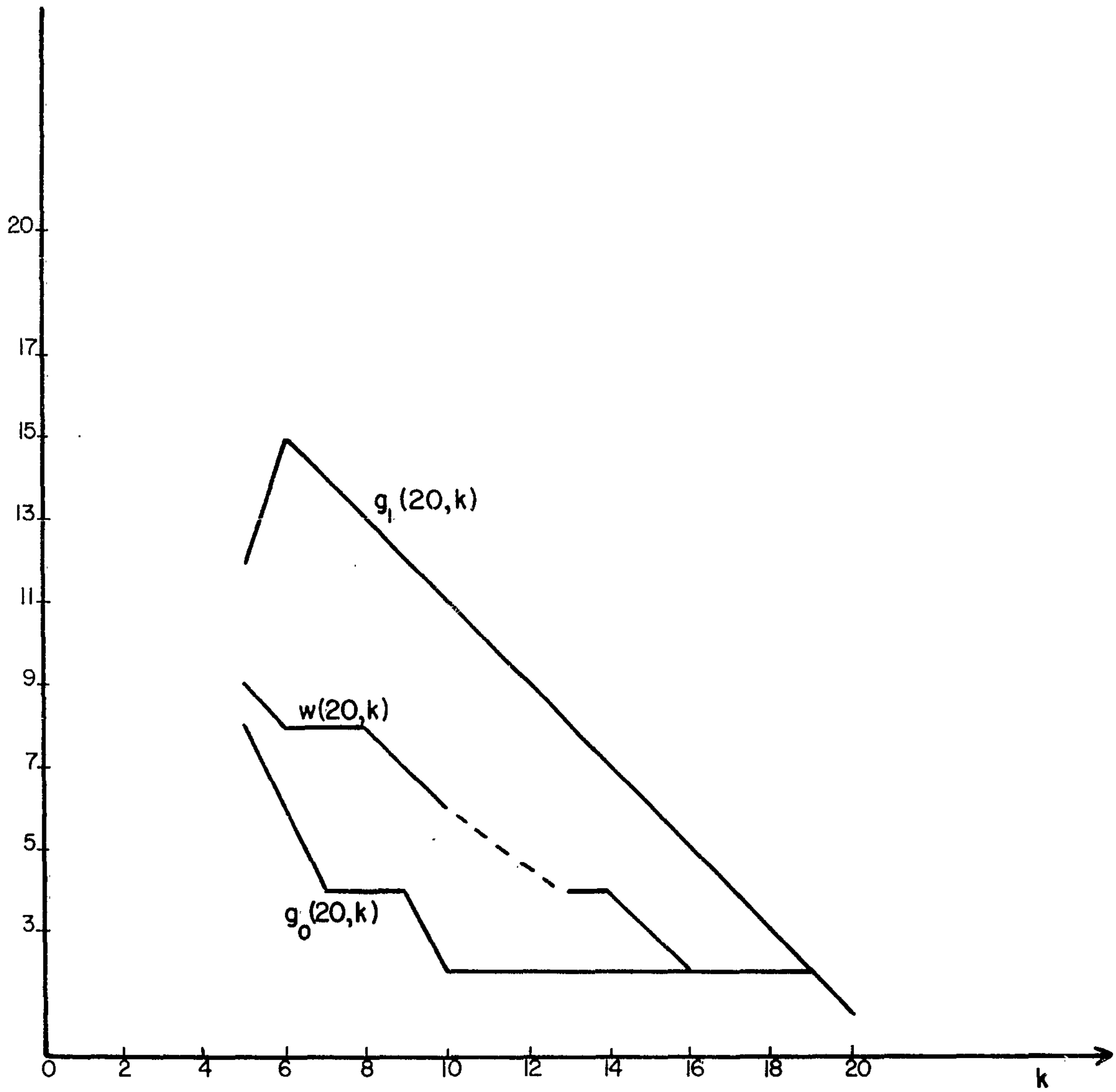
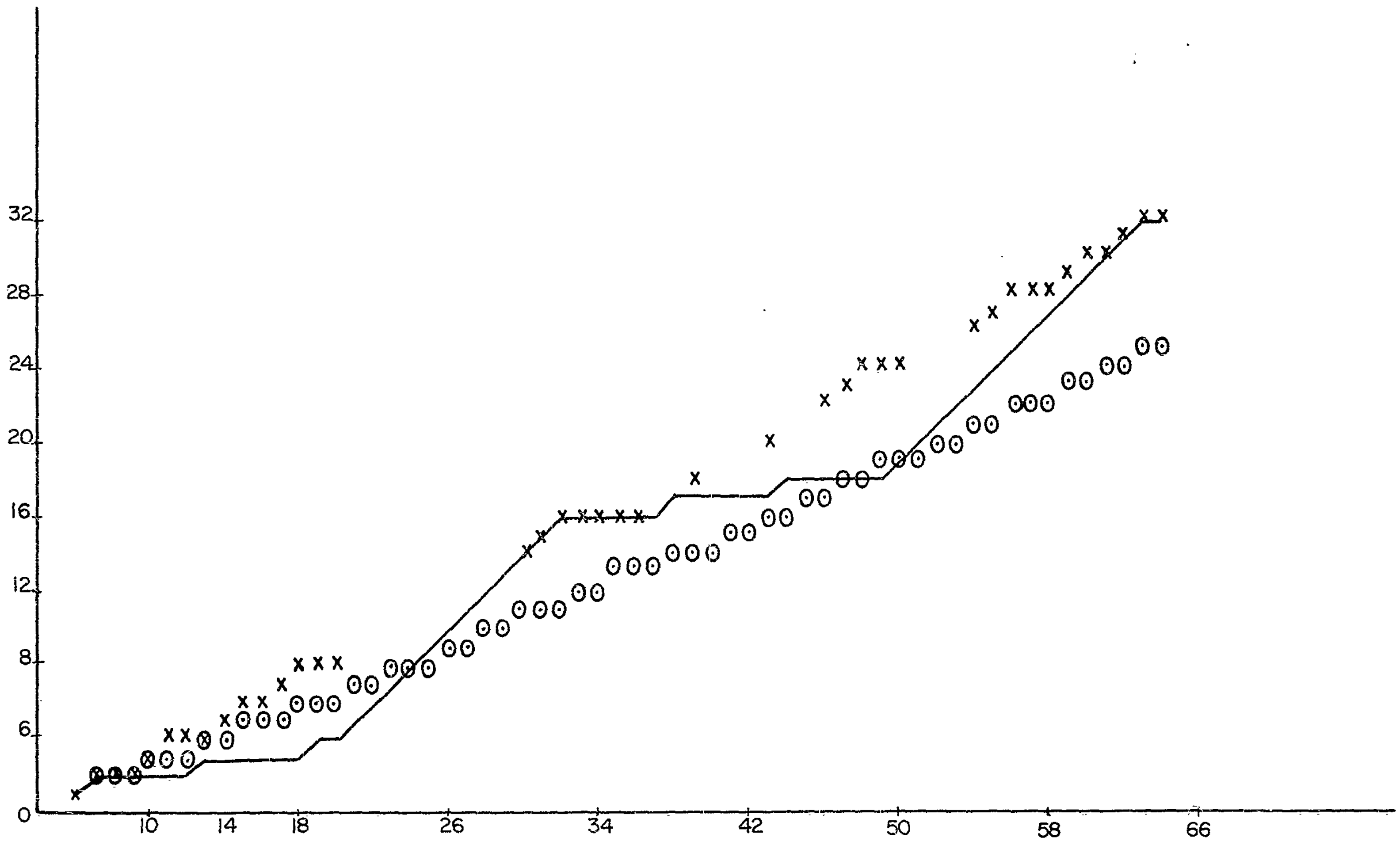
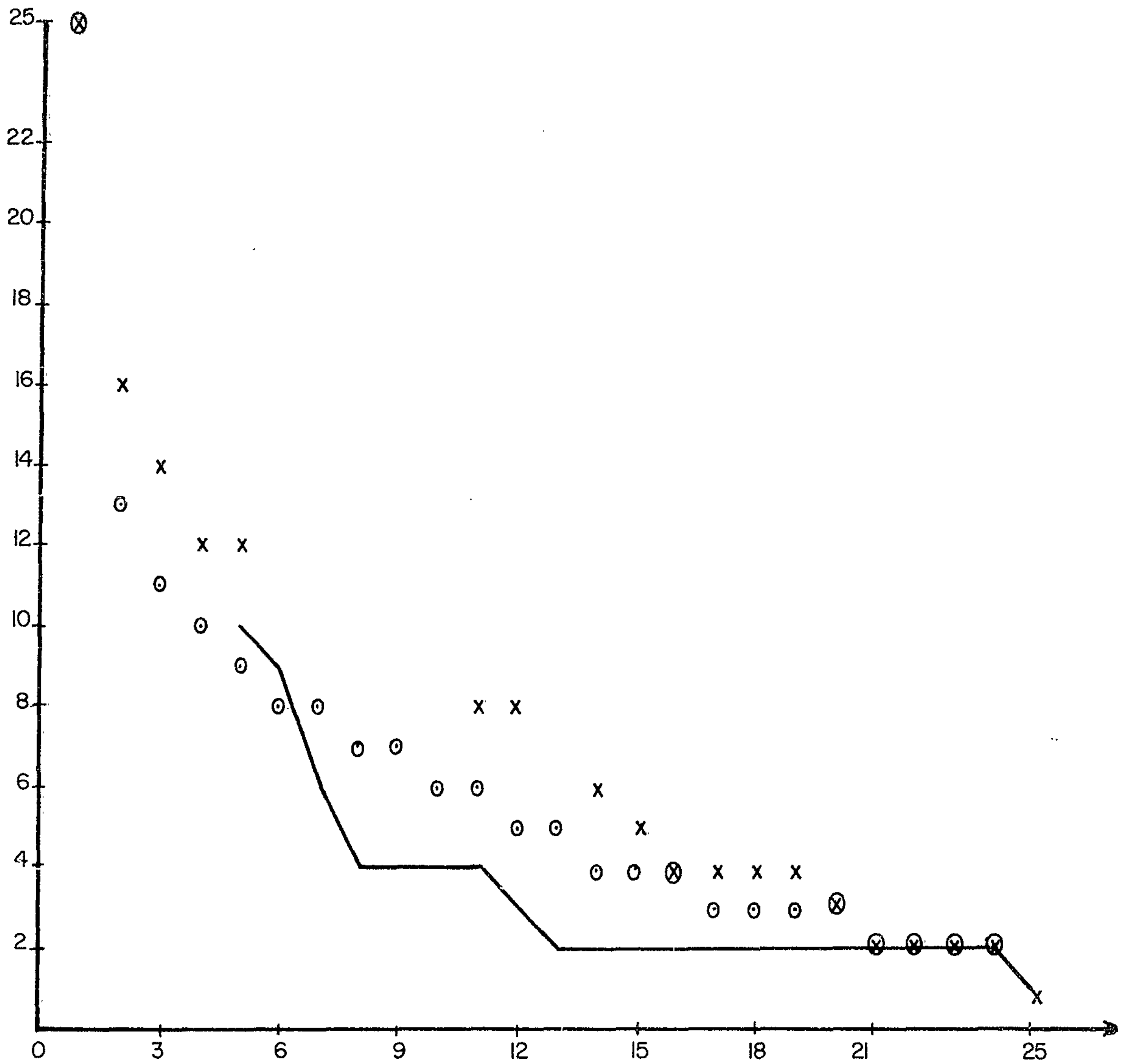


Fig. 4



Comparison of g_o with the Varsharmov-Gilbert bound (O) for k= 6
(known values of w a e marked by x)

Fig. 5



Comparison of g_o with the Varsharmov-Gilbert bound $\odot$ for n=25

(known values of w are marked by x.)

Fig. 6

Difference: ($\mathcal{J}_0$ - Varsharmov)

n \ k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	0															
3		0														
4		0	0													
5			0	0												
6			0	0	0											
7			1	0	0	0										
8			0	1	0	0	0									
9				1	-1	0	0	0								
10				0	-1	-1	0	0	0							
11				0	-1	-1	-1	0	0	0						
12				0	0	-1	-1	-1	0	0	0					
13				1	1	-1	-1	-1	-1	0	0	0				
14				2	1	0	-2	-1	-1	-1	0	0	0			
15				2	2	-1	-1	-2	-1	-1	-1	0	0	0		
16				2	2	-1	0	-2	-2	-1	-1	0	0	0	0	
17					2	-1	-1	-1	-2	-1	-1	-1	0	0	0	0
18					2	-2	-1	-1	-2	-2	-1	-1	-1	0	0	0
19					1	-2	-2	-1	-2	-2	-2	-1	-1	-1	0	0
20					1	-1	-2	-1	-1	-2	-2	-2	-1	-1	-1	0
21					1	-1	-2	-2	-1	-2	-2	-2	-2	-1	-1	-1
22					2	-1	-3	-2	-2	-1	-3	-2	-2	-2	-1	-1
23					2	-1	-2	-2	-2	-1	-2	-3	-2	-2	-2	-1
24					1	0	-1	-3	-2	-2	-1	-3	-2	-2	-2	-1
25					1	1	-2	-3	-3	-2	-2	-2	-3	-2	-2	-2
26					1	1	-2	-3	-3	-2	-2	-2	-3	-3	-2	-2
27					2	2	-3	-2	-3	-3	-2	-2	-2	-3	-3	-2
28					2	2	-3	-2	-4	-3	-3	-2	-2	-3	-3	-3
29					3	3	-3	-3	-3	-4	-3	-3	-2	-3	-3	-3
30					4	3	-4	-3	-2	-4	-3	-3	-2	-2	-4	-3
31					4	4	-3	-3	-3	-4	-4	-3	-3	-2	-3	-4
32					4	5	-3	-4	-3	-4	-4	-4	-3	-3	-2	-4

Fig. 7

3. Study in the region $k \leq n \leq 2^{k-1}$

From Corollary 3. to Proposition 7. of [4] follows that w is one of the functions f verifying:

$$19.\quad f(n,k) \leq \left[\frac{n-1}{2}\right] \text{ if } n < 2^{k-1}.$$

The use of this relation will not improve the lower bound g_0, but has some effect on the upper bound. Let g_2 be defined for $0 < k \leq n \leq 2^{k-1}$ by:

$$g_2(1,1) = 1$$
$$g_2(n,k) = g_2(n-1,\ k-1) \qquad \text{if} \qquad n \leq 2^{k-2}$$
$$g_2(n,k) = \begin{cases} 2h-1 & \text{if} \quad n = 4h \\ 2h & \text{if} \quad n = 4h+1,\ 4h+2,\ 4h+3 \end{cases} \Bigg\}\ 2^{k-2} < n < 2^{k-1}-1$$
$$g_2(2^{k-1}-1,\ k) = 2^{k-2}-1$$
$$g_2(2^{k-1},\ k) = 2^{k-2}.$$

Proposition 15 Let f satisfy 1. to 19.: then

$$20.\quad f(n,k) \leq g_2(n,k) \qquad \text{if} \quad 0 < k \leq n \leq 2^{k-1}.$$

The proof is by induction on k. For $k=1$, then $n \leq 2^{k-1} = 1$ and 20. holds by definition of g_2. So assume 20. for $k-1$. By 3. we have, for $n \leq 2^{k-2}$,

$$f(n,k) \leq f(n-1,k-1) \leq g(n-1,\ k-1) = g(n,k).$$

Thus assume $n > 2^{k-2}$. Clearly, because of 15. and 19., we can also assume $n < 2^{k-1}-1$. In this interval, then, 20. follows at once from 5. and 19. In fact $g_2(n,k) = \left[\frac{n-1}{2}\right]$ if $n = 4h,\ 4h+1,\ 4h+2$; thus we could only have $f(4h+3,\ k) = g_2(4h+3,\ k)+1 = 2h+1$: but then

$$f(4h+4,\ k) = 2h+2 > \left[\frac{n-1}{2}\right],$$

a contradiction.

Fig. 8 and 9 illustrate the relations between the five bounds f_0, f_1, g_0, g_1, g_2. Crosses locate known values of w. These graphs suggest that g_2 is a rather good upper bound for w, whereas g_0 should be further improved. In particular, fig. 8 as well as fig. 5 show possibly that the knowledge of $w(2^{k-2}, k)$, or a good lower bound for it, would considerably improve g_0.

Fig. 10, 11, and 12 compare with Hamming's and Plotkin's upper bounds. To compute g_2 for a given pair (n, k) with $n \leq 2^{k-2}$, determine an integer $r > 0$ such that

$$2^{k-r-2} < n-r < 2^{k-r-1} .$$

Then $g_2(n, k) = g_2(n-r, k-r)$ is explicitly given in the definition. It may also be appropriate to observe that $g_2(n, k+r) = g_2(n-r, k)$ for all $r \geq 0$: in fact g_2 is constant for constant $n-k$.

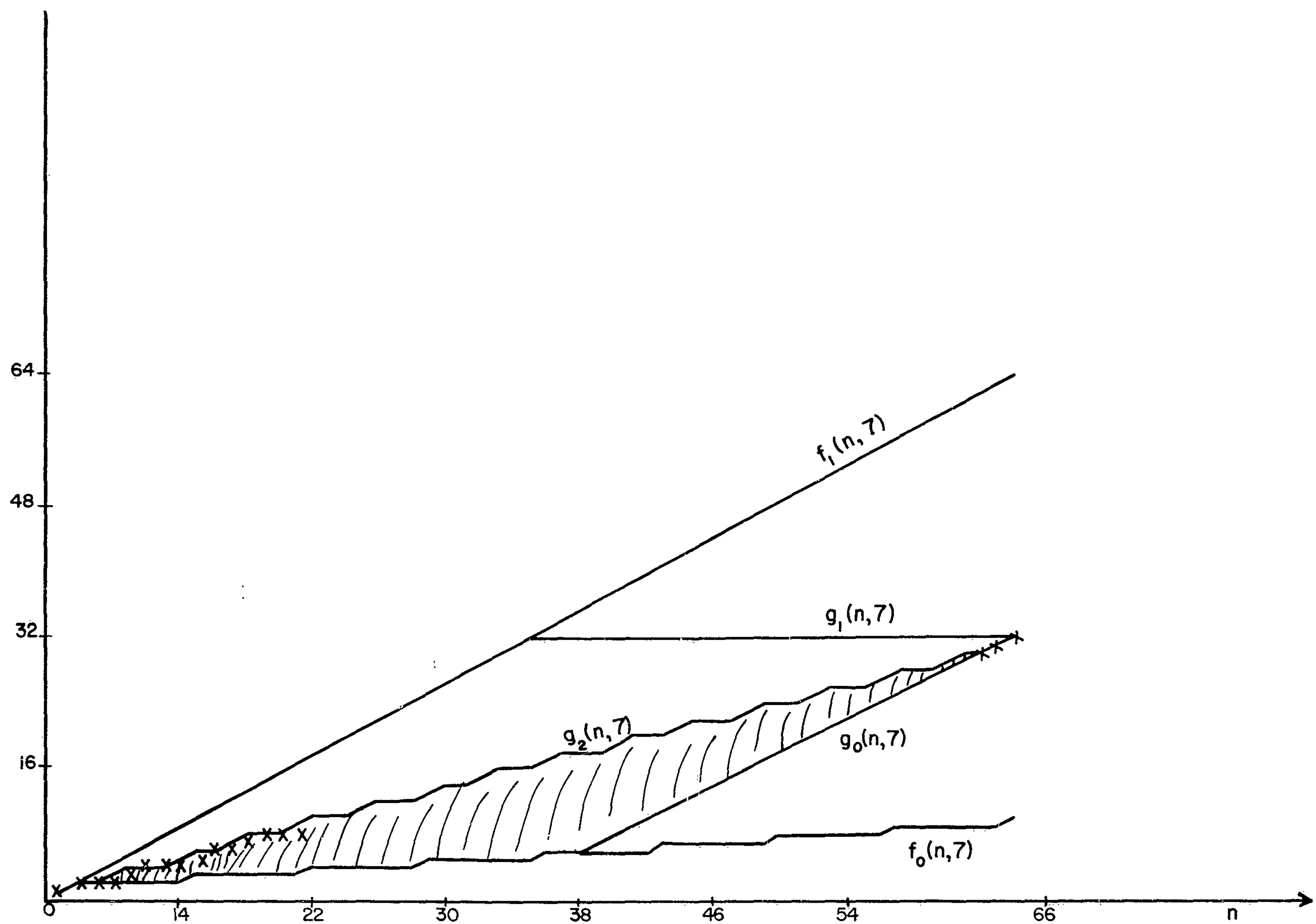
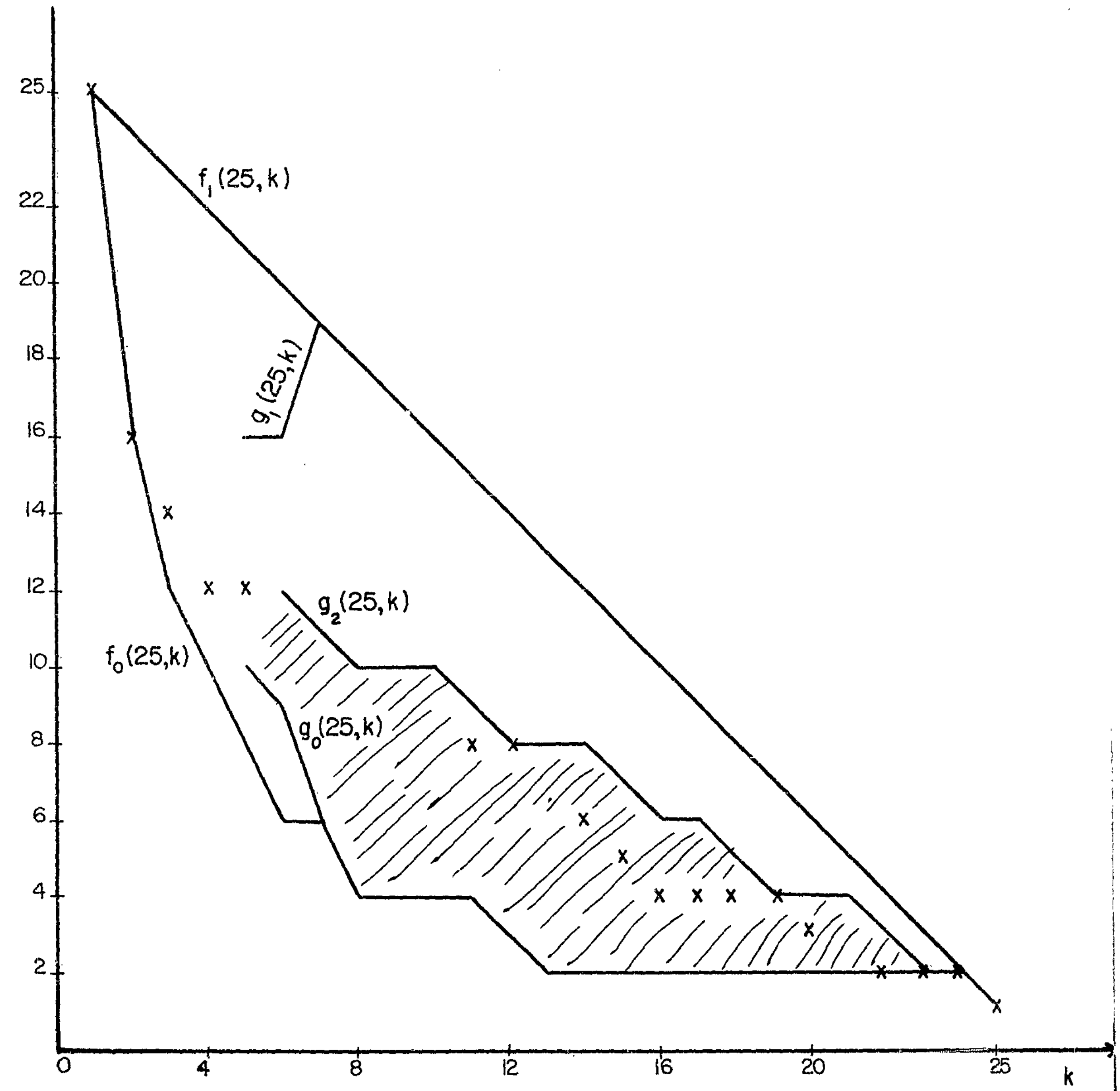
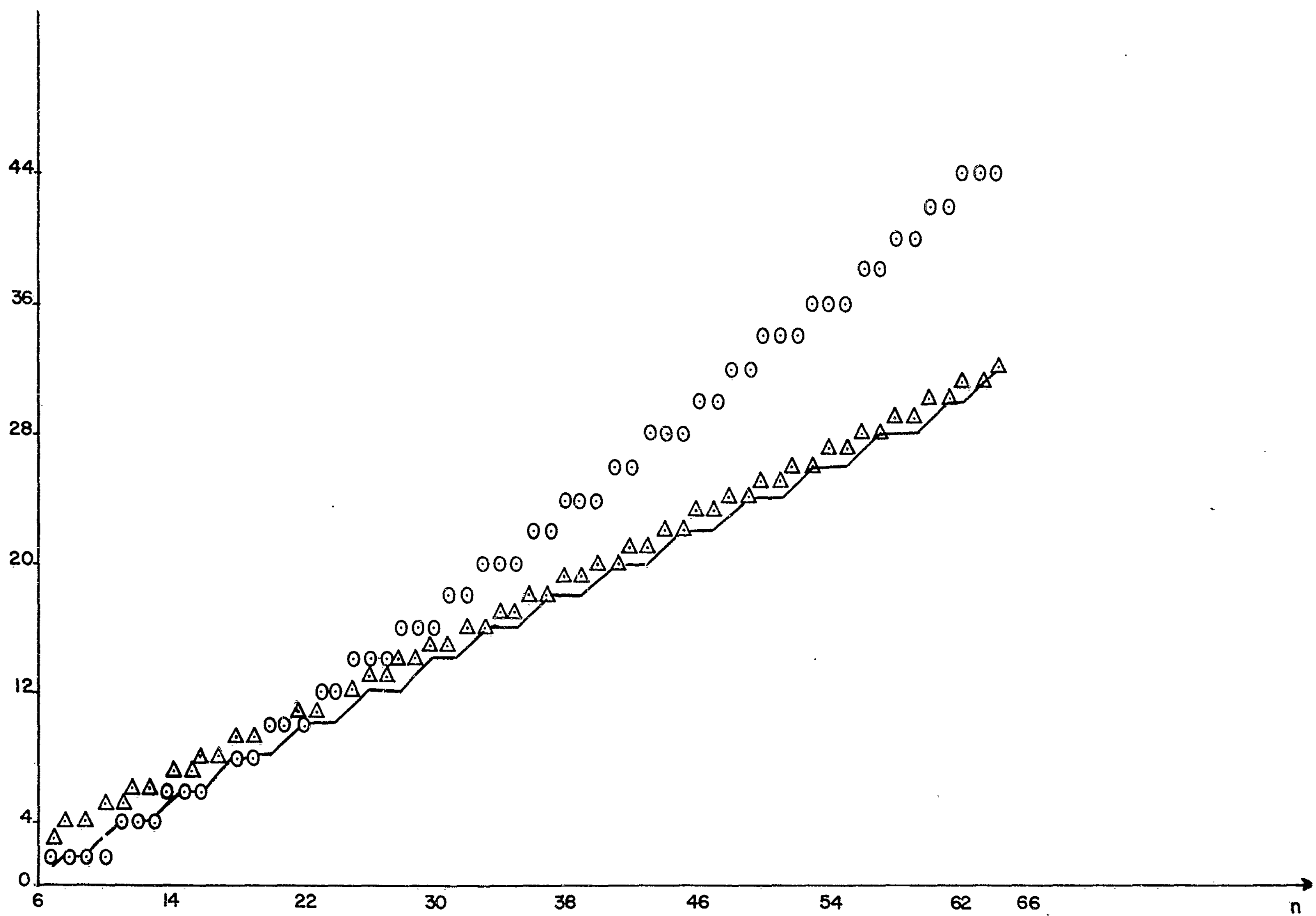


Fig. 8



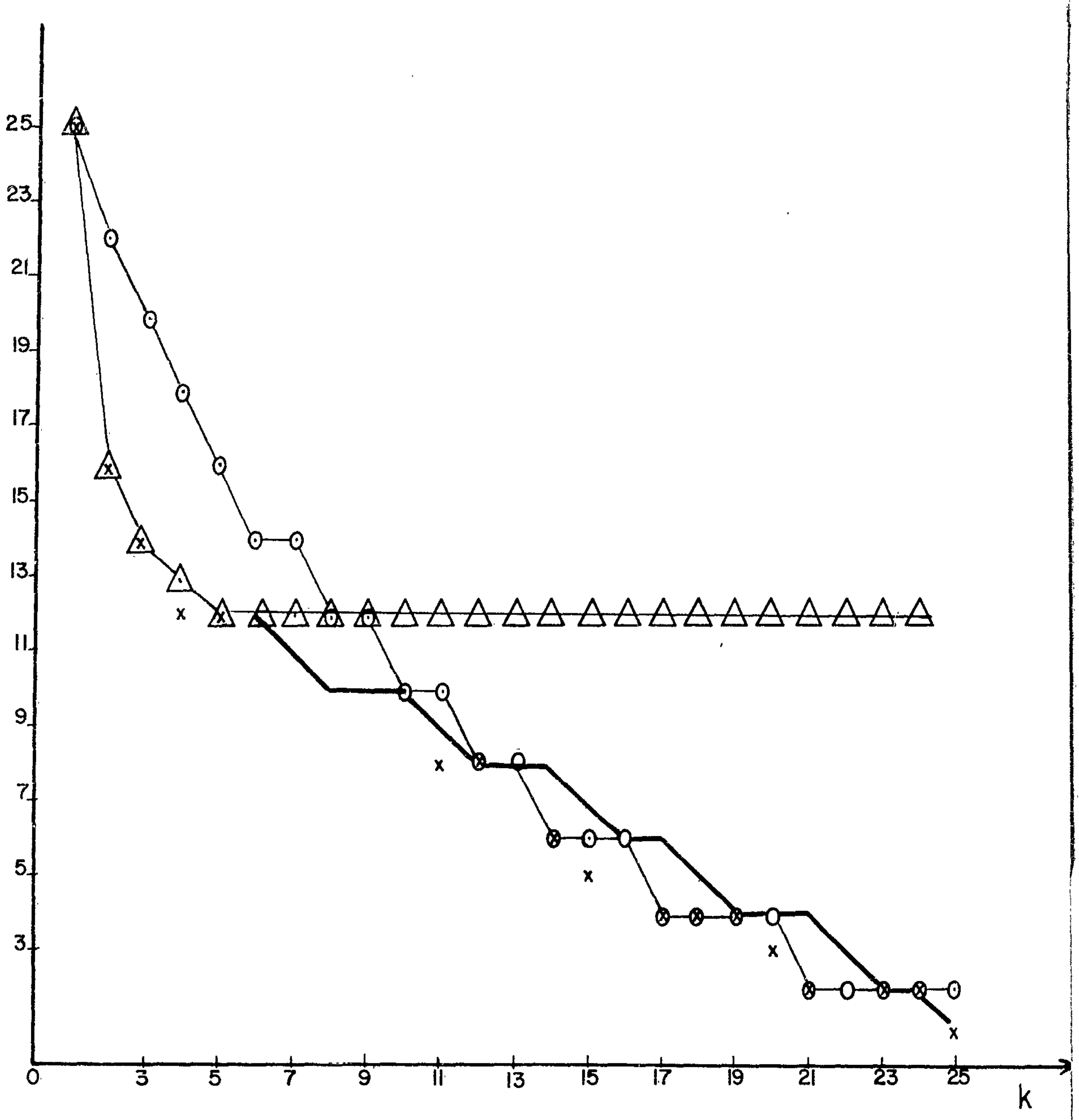
$f_1(25,k)$
$g_1(25,k)$
$g_2(25,k)$
$f_0(25,k)$
$g_0(25,k)$
25
22
20
18
16
14
12
10
8
6
4
2
0
4
8
12
16
20
25
k

Fig. 9



Comparison of g_2, Hammings ⊙ and Plotkin △ for $k = 7$

Fig. 10



Comparison of g_2, Hamming ⊙, and Plotkin △ for $n = 25$
(known values of w are denoted by x)

Fig. 11

Difference: (Hamming - g_2)

n \ k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
1	1																										
2		1																									
3			1																								
4			0	1																							
5				0	1																						
6				0	0	1																					
7				1	0	0	1																				
8				0	-1	0	0	1																			
9					0	-1	0	0	1																		
10					0	0	-1	0	0	1																	
11					0	0	0	-1	0	0	1																
12					1	0	0	0	-1	0	0	1															
13					0	1	0	0	0	-1	0	0	1														
14					2	0	1	0	0	0	-1	0	0	1													
15					1	0	0	1	0	0	0	-1	0	0	1												
16					0	1	0	0	-1	0	0	-2	-1	0	0	1											
17						0	1	0	0	-1	0	0	-2	-1	0	0	1										
18						2	0	1	0	0	-1	0	0	-2	-1	0	0	1									
19						2	0	0	-1	0	0	-1	0	0	-2	-1	0	0	1								
20						1	2	0	0	-1	0	0	-1	0	0	-2	-1	0	0	1							
21						2	1	2	0	0	-1	0	0	-1	0	0	-2	-1	0	0	1						
22						2	0	1	0	0	0	-1	0	0	-1	0	0	-2	-1	0	0	1					
23						2	2	0	1	0	0	0	-1	0	-2	-1	0	0	-2	-1	0	0	1				
24						3	2	2	0	1	0	0	-2	-1	0	-2	-1	0	0	-2	-1	0	0	1			
25						2	3	2	0	0	1	0	0	-2	-1	0	-2	-1	0	0	-2	-1	0	0	1		
26						4	2	1	2	0	0	-1	0	0	-2	-1	0	-2	-1	0	0	-2	-1	0	0	1	
27						4	2	2	1	2	0	0	-1	0	0	-2	-1	0	-2	-1	0	0	-2	-1	0	0	1
28						3	4	2	2	1	2	0	0	-1	0	0	-2	-1	0	-2	-1	0	0	-2	-1	0	0
29						4	3	2	2	0	1	0	0	0	-1	0	0	-2	-1	0	-2	-1	0	0	-2	-1	0
30						4	2	3	2	2	0	1	0	0	0	-1	0	-2	-2	-1	0	-2	-1	0	0	-2	-1
31						5	4	2	3	2	2	0	1	0	0	-2	-1	0	-2	-2	-1	0	-2	-1	0	0	-2
32						4	3	4	2	1	2	0	0	1	0	0	-2	-1	0	-2	-2	-1	-2	-2	-1	0	-2

Fig. 12

4. Some scattered values of w.

We will denote by P the Plotkin bound:

$$P(n,k) = \left\lceil \frac{n 2^{k-1}}{2^k - 1} \right\rceil .$$

Proposition 16 For any meaningful value of m, we have:

$$\left.\begin{aligned} w(m(2^k-1)-2,k) &= P(m(2^k-1)-2;k) = m2^{k-1}-2 \\ w(m(2^k-1)-1,k) &= P(m(2^k-1)-1,k) = m2^{k-1}-1 \\ w(m(2^k-1),k) &= P(m(2^k-1),k) = m2^{k-1} \\ w(m(2^k-1)+1,k) &= P(m(2^k-1)+1,k) = m2^{k-1} \end{aligned}\right\} k \geq 2$$

$$w(m(2^k-1)+2,k) = P(m(2^k-1)+2,k)-1 = m2^{k-1} \qquad k \geq 3$$

$$w(m(2^k-1)+3,k) = P(m(2^k-1)+3,k)-1 = m2^{k-1} \qquad k \geq 4.$$

Set $n = m(2^k-1)$; then $w(n,k) = P(n,k)$ is our relation 13. and has been established, e.g., in [5,2]. The equalities for $n-1$ and $n-2$ follow then from the inequality $w \leq P$ and from 4. Similarly, the result for $n+1$ follows from 2.

We have now

$$m2^{k-1} \leq w(n+2,k) \leq m2^{k-1}+1 = P(n+2,k).$$

If $w(n+2,k)$ has the larger of the two possible values, there is a code $A(n+2,k)$ with elements x_i , $i = 0,1,\ldots,2^k-1$ whose weights w_i verify $w_0 = 0$ and

$$w_i = m2^{k-1}+1 + d_i \qquad d_i \geq 0, \quad i = 1,2,\ldots,2^k-1.$$

Then $\sum w_i = (n+2)2^{k-1} = (m2^{k-1}+1)(2^k-1)+\sum d_i$ yielding $\sum d_i = 1$ and thus $d_i \neq 0$ for exactly one subscript. From [4, Prop. 4] it follows then that $m2^{k-1}+1+1$ has to be a non-negative multiple of 2^{k-1} or $2 = h2^{k-1}$ and then $k \leq 2$. If $k \geq 3$ then, $w(n+2,k) = m2^{k-1}$. For $n+3$ the proof is similar: we obtain, if $w(n+3,k) = P(n+3,k)$

$$\sum w_i = (n+3)2^{k-1} = (m2^{k-1}+1)(2^k-1) + \sum d_i$$

or $\sum d_i = 2^{k-1}+1$. Hence $m2^{k-1}+1 = \sum d_i + (m-1)2^{k-1}$:

but [4, Prop. 6] requires $m-1<0$, or $m=0$. This implies $n+3=3$ and hence $k\leq 3$.

Proposition 15 is thus established. Notice that the techniques used to prove it yield also further, but less precise, results. For instance, for $k\geq 4$.

$$w(m(2^k-1)+4,\ k)\leq P(m(2^k-1)+4,\ k)-1=m2^{k-1}+1.$$

Proposition 17 For $m\geq 0,\ 0<r<k$, and $n=m(2^k-1)+\sum_{i=r}^{k-1}2^i$, we have

$$w(n-2,\ k)=P(n-2,\ k)-1=\frac{n+m}{2}-2 \quad \text{if} \quad n+m=4\ell$$

$$w(n-1,\ k)=P(n-1,\ k)=\frac{n+m}{2}-1$$

$$w(n,\ k)=P(n,k)=\frac{n+m}{2}$$

$$w(n+1,\ k)=\frac{n+m}{2}=\begin{cases}P(n+1,\ k) & r>1\\ P(n+1,\ k)-1 & r=1.\end{cases}$$

Notice that $\frac{n+m}{2}=m2^{k-1}+\sum_{i=r}^{k-1}2^{i-1}$. The first three equalities have been established in [2] (see also [5,6,7]); they could be obtained also with the tools developed here and in [4]. To prove the last equality observe that, if $r>1$,

$$\frac{n+m}{2}=w(n,k)\leq w(n+1,\ k)\leq P(n+1,\ k)=\frac{n+m}{2}.$$

If $r=1$, then $n+1=m(2^k-1)+2^k-2=(m+1)(2^k-1)-1$ and our result follows from Prop. 16.

Corollary For $k\geq 4$,

$$w(2^{k-1}+2,\ k)=P(2^{k-1}+2,\ k)-1=2^{k-2}$$

$$w(2^{k-1}+3,\ k)=P(2^{k-1}+3,\ k)-1=2^{k-2}.$$

In fact, using Prop. 16:

$$2^{k-2}=w(2^{k-1}+1,\ k)\leq w(2^{k-1}+2,\ k)\leq w(2^{k-1}+3,\ k)\leq$$
$$\leq w(2^{k-1}+2,\ k-1)=2^{k-2}.$$

5. The function w for $k \leq 4$.

The aim of this section is to prove the following result, that can be found also in [8]:

Proposition 18 For $k \leq 4$, w is given by the table of fig. 13 together with

21. $$w(n+2^{k}-1, k) = w(n,k) + 2^{k-1}.$$

Equivalently, we can describe w as follows:

Corollary 1 We have $w(n,1) = n$ and $w(n,2) = \left[\frac{2n}{3}\right]$.

Further:

$$\left.\begin{array}{ll} w(7m,3) = 4m & w(15m,4) = 8m \\ w(7m+1,3) = 4m & w(15m+1,4) = 8m \\ w(7m+2,3) = 4m & w(15m+2,4) = 8m \\ & w(15m+3,4) = 8m \end{array}\right\} m \geq 1$$

$$\left.\begin{array}{ll} w(7m+3,3) = 4m+1 & w(15m+4,4) = 8m+1 \\ w(7m+4,3) = 4m+2 & w(15m+5,4) = 8m+2 \\ w(7m+5,3) = 4m+2 & w(15m+6,4) = 8m+2 \\ w(7m+6,3) = 4m+3 & w(15m+7,4) = 8m+3 \\ & w(15m+8,4) = 8m+4 \\ & w(15m+9,4) = 8m+4 \\ & w(15m+10,4) = 8m+4 \\ & w(15m+11,4) = 8m+5 \\ & w(15m+12,4) = 8m+6 \\ & w(15m+13,4) = 8m+6 \\ & w(15m+14,4) = 8m+7 \end{array}\right\} m \geq 0$$

Corollary 2 For $k = 1,2$, $w(n,k) = P(n,k)$. Furthermore $w(n,3) = P(n,3)$ if $n \neq 7m+2$ and $w(7m+2,3) = P(7m+2,3) - 1$; $w(n,4) = P(n,4)$ if $n \neq 15m+2,\ 15m+3,\ 15m+4,\ 15m+6,\ 15m+10$, and in these cases $w(n,4) = P(n,4) - 1$.

n \ k	1	2	3	4
1	1			
2	2	1		
3		2	1	
4		2	2	1
5			2	2
6			3	2
7			4	3
8			4	4
9			4	4
10				4
11				5
12				6
13				6
14				7
15				8
16				8
17				8
18				8

Fig. 13

This corollary follows at once from the observation that

$$P(n+2^{k}-1,\ k) = P(n, k) + 2^{k-1}.$$

Finally it may be interesting to point out that the following formula yields ω for $k \leq 4$. Set

$$n-k+1 = \sum_{\ell=1}^{k} (2^{\ell}-1)\, a_{\ell}$$

where a_k is the largest integer such that $(2^{k}-1)a_k \leq n-k+1$, and a_{ℓ}, $\ell < k$ is the largest integer with $(2^{\ell}-1)a_{\ell} \leq n-k+1-\sum_{\ell+1}^{k}(2^{i}-1)\, a_i$; then:

<u>Corollary 3</u> For $k \leq 4$, we have

$$\omega(n, k) = \sum_{\ell=1}^{k} 2^{\ell-1} a_{\ell} = 1/2\ \left(n-k+1+\sum_{i=1}^{k} a_i\right).$$

This corollary is obtained by verifying first the values of the table, and then relation 21. To establish the proposition itself, we shall actually prove directly the Corollary 1. For $k=1$, we use relation 9. For $k=2$, it is easy to construct codes $A(n,2)$ with $\omega(A) = \left\lfloor \frac{2n}{3} \right\rfloor$; on the other hand $\omega(n,2) \leq P(n,2) = \left\lfloor \frac{2n}{3} \right\rfloor$.

For $k=3$, Corollary 1 is obtained by setting $k=3$ in Prop. 16 and Prop. 17. If we do the same for $k=4$ we obtain all the desired relations, but those for $15m+4$ and $15m+5$. These are obtained immediately by observing that $\omega(4,4)=1$ and thus

$$8m+1 \leq \omega(15m+4,\ 4) \leq \omega(15m+3,\ 4)+1 = 8m+1$$

giving $\omega(15m+4, 4) = 8m+1$ and, by 5., $\omega(15m+5,\ 4) = 8m+2$.

It has been assumed in two papers [9, 10] that relation 21 holds for all k. <u>This is false</u>. We give here a counterexample to show that, in particular, 21. does not hold for $k=5$. Let $n=8$ and $k=5$. If 21 holds, $\omega(39, 5) = \omega(8,5)+16$. Observe that $\omega(8,5) = 2$ (this follows in particular from Prop. 20, to be proven later). Thus 21 implies that $\omega(39,5) = 2+16 = 18$.

In [5] and [2] it has been shown that $w(24,5)=12$; and in [11] there is a code showing $w(15,5)\geq 7$. Hence, by relation 4 and Proposition 1, $w(39,5)\geq 7+12 = 19$, contradicting 21.

It is important to note that the periodicity suggested by 21 <u>does</u> hold for <u>sufficiently large n</u>. More precisely, we have:

<u>Proposition 19</u>

To every k there corresponds an integer $n_0(k)$ such that for any $n\geq n_0(k)$,

$$w(n+h\cdot(2^k-1),k)=w(n,k)+h\cdot(2^{k-1}),\quad h=1,2,3,\ldots .$$

From relation 4 and Propositions 1 and 16, we have, for any n, h and k :

$$w(n+(h+1)(2^k-1),k)\geq w(n+h(2^k-1),k)+2^{k-1}.$$

Setting $q(n,h,k)=P(n+h(2^k-1),k)-w(n+h(2^k-1),k)\geq 0$, and remembering that $P(n+(h+1)\cdot(2^k-1),k)=P(n+h\cdot(2^k-1),k)+2^{k-1}$, we obtain:

$$w(n+(h+1)\cdot(2^k-1),k)+q(n,h+1,k)=w(n+h\cdot(2^k-1),k)+2^{k-1}+q(n,h,k).$$

Therefore, $q(n,h+1,k)\leq q(n,h,k)$.

Since q is non-negative, non-increasing, and an integer, there exists an integer $h_0(n,k)$ such that $q(n,h+1,k)=q(n,h,k)$ for all $h\geq h_0(n,k)$.

Since any n can be written in the form $n=n'+h\cdot(2^k-1)$, where $0\leq n'<2^{k-1}$, it is sufficient to consider n only in this range. Now let $N(k)$ be the set consisting of the 2^k-1 integers of the form

$$n+h_0(n,k)\cdot(2^k-1),\qquad \text{for } 0\leq n<2^k-1.$$

The largest element of $N(k)$ is then the $n_0(k)$ in Proposition 19.

Proposition 18 merely states that $n_0(k)=k$ for $k=1,2,3$, and 4 .

<u>6. The region in which $w(n,k)\leq 4$</u>.

We shall prove here

<u>Proposition 20</u>. The following relations hold:

a. $w(n,k)=2$ if and only if $2^h\leq n<2^{h+1}$ and $1\leq n-k\leq h$ for some integer h ;

b. $w(n,k)=3$ if and only if $2^h < n < 2^{h+1}$ and $n-k=h+1$ for some integer h;

c. $w(n,k)=4$ if, for some integer h

$\alpha)$ $2^h=n$ and $h+1 \leq n-k \leq 2h-1$,

or $\beta)$ $2^h < n < 2^{h+1}$ and $h+2 \leq n-k \leq 2h-1$,

or $\gamma)$ $6 \leq 2^h+2^{h-1} \leq n < 2^{h+1}$ and $n-k=2h$.

Since, for instance, $w(7,2)=4$, we do not have "only if" in part c. The proof will be based upon Hamming bound $H(n,k)=2e+2$ where e is the largest integer such that $\sum_{r=0}^{e}\binom{n}{i} \leq 2^{n-k}$. Proposition 6. and its Corollary imply $w(n,n-1)=2$ and thus $w(n,k) \geq 2$ if $k \leq n-1$ or $1 \leq n-k$. On the other hand, under the assumptions of a., we have

$$2^{n-k} \leq 2^h < 1+n,$$

that is $w(n,k) \leq H(n,k)=2$. Thus the "if" part of a. is established.

Let us now prove c.α) by first observing that, if $n=2^h$, there is a Reed-Muller code A with $k=2^h-h-1$ and $4=2^2=w(A) \leq w(n,k)$.

On the other hand

$$1+n+\binom{n}{2}=1+2^{h-1}+2^{2h-1} > 2^{2h-1}$$

and thus $H(n,k) \leq 4$ as long as $n=2^h$ and $n-k \leq 2h-1$. We can now prove the "if" part of b. If n and k verify the assumptions of b., then $n-1$ and k verify the assumptions of a., and thus $w(n-1,k)=2$ or $2 \leq w(n,k) \leq 3$. If now $n=2^{h+1}-1$, by c.α) $w(n+1,k)=4$ and thus $w(n,k)=3$. If $n<2^{h+1}-1$ we apply 3.: $w(n,k)=w(n,n-h-1) \geq$ $\geq w(2^{h+1}-1, 2^{h+1}-1-h-1)=3$. Having thus obtained the "if" part of b., Proposition 6 gives the "only if" part of a. and 5. gives the "only if" part of b.

Relation 5. yields also, because of b., $w(n,k)=4$ if $2^h < n < 2^{h+1}$ and $n-k=h+2$, which is part of c.β). For the larger values of $n-k$ considered in β) it is easy to show $H(n,k) \leq 4$.

This last inequality holds true also under the assumption of γ). Under those assumptions we have

$$w(n,k)=w(n,n-2h) \geq w(n,n-2h+1)=4$$

by β). This completes the proof of our proposition. The knowledge of more accurate bounds for w would enable us perhaps to extend these results to reach regions of more practical interest.

7. A small table of values

We give here two tables of values of w that have been found using the results given above, those of [4], known codes and several well known "tricks". The tables cover the following ranges:

$$1 \leq k \leq 6, \quad k \leq n \leq 100$$

$$1 \leq k \leq n \leq 24.$$

The last is illustrated in fig. 16 and 17. In the relatively few instances in which the exact value of w is still unknown, lower and upper bounds are given.

1	2	3	4	5	6
1					
2	1				
3	2	1			
4	2	2	1		
5	3	2	2	1	
6	4	3	2	2	1
7	4	4	3	2	2
8	5	4	4	2	2
9	6	4	4	3	2
10	6	5	4	4	3
11	7	6	5	4	4
12	8	6	6	4	4
13	8	7	6	5	4
14	9	8	7	6	5
15	10	8	8	7	6
16	10	8	8	8	6
17	11	9	8	8	7
18	12	10	8	8	8
19	12	10	9	8	8
20	13	11	10	9	8
21	14	12	10	10	8-1
22	14	12	11	10	9-10
23	15	12	12	11	9-10
24	16	13	12	12	9-11
25	16	14	12	12	10-12
26	17	14	13	12	10-12
27	18	15	14	13	11-12
28	18	16	14	14	12-13
29	19	16	15	14	13-14
30	20	16	16	15	14
31	20	17	16	16	15
32	21	18	16	16	16
33	22	18	16	16	16

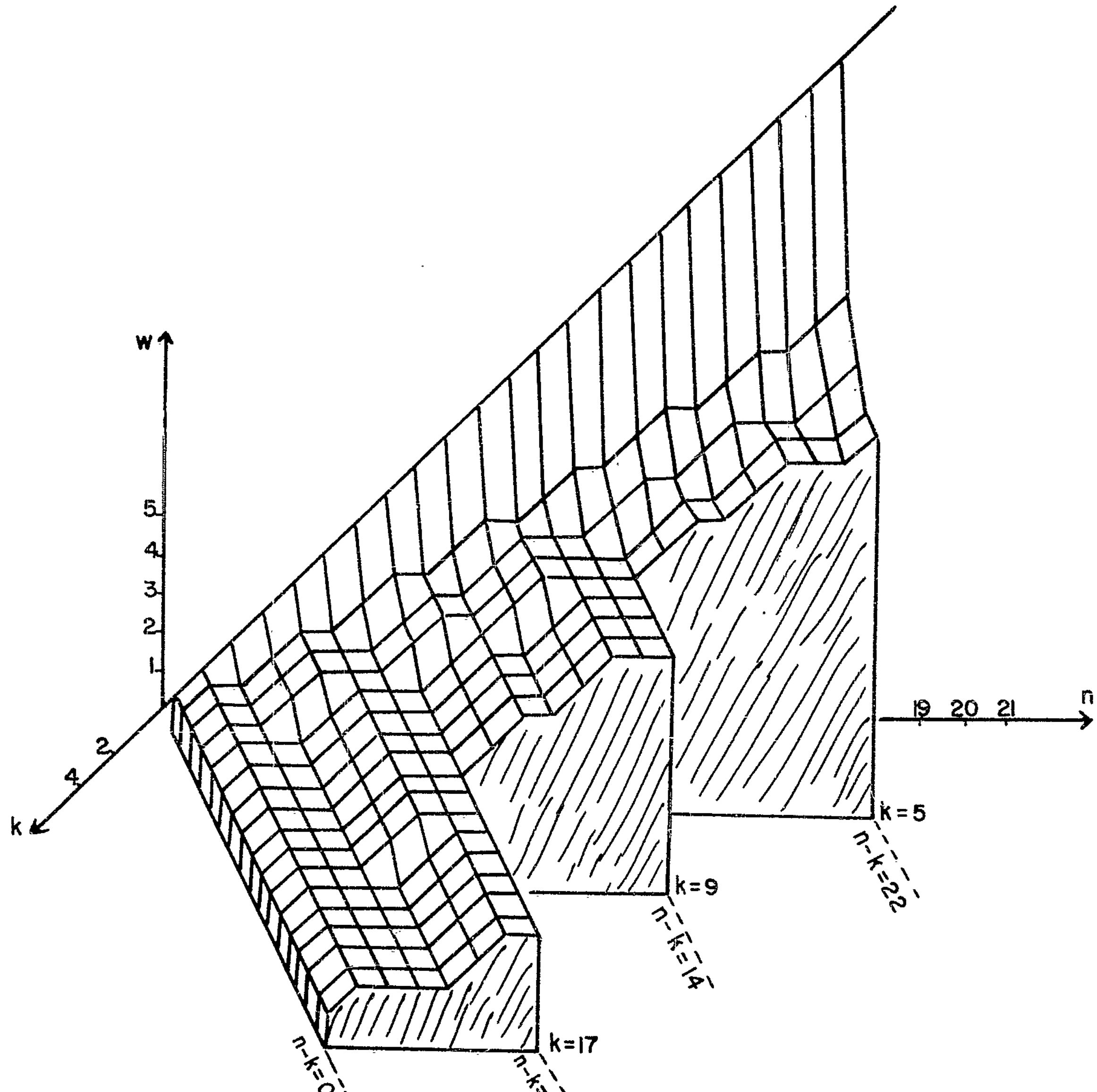
1	2	3	4	5	6
34	22	19	17	16	16
35	23	20	18	16	16
36	24	20	18	17	16
37	24	20	19	18	16-17
38	25	21	20	18	17-18
39	26	22	20	19	18
40	26	22	20	20	18-19
41	27	23	21	20	19-20
42	28	24	22	20	19-20
43	28	24	22	21	20
44	29	24	23	22	20-21
45	30	25	24	22	21-22
46	30	26	24	23	22
47	31	26	24	24	23
48	32	27	24	24	24
49	32	28	25	24	24
50	33	28	26	24	24
51	34	28	26	25	24-25
52	34	29	27	26	24-26
53	35	30	28	26	25-26
54	36	30	28	27	26
55	36	31	28	28	27
56	37	32	29	28	28
57	38	32	30	28	28
58	38	32	30	29	28
59	39	33	31	30	29
60	40	34	32	30	30
61	40	34	32	31	30
62	41	35	32	32	31
63	42	36	32	32	32
64	42	36	33	32	32
65	43	36	34	32	32
66	44	37	34	33	32

1	2	3	4	5	6
67	44	38	35	34	32-33
68	45	38	36	34	32-34
69	46	39	36	34	32-34
70	46	40	36	35	33-34
71	47	40	37	36	34-35
72	48	40	38	36	34-36
73	48	41	38	36	35-36
74	49	42	39	37	36
75	50	42	40	38	36-37
76	50	43	40	38	36-37
77	51	44	40	39	37-38
78	52	44	40	40	38
79	52	44	41	40	39
80	53	45	42	40	40
81	54	46	42	41	40
82	54	46	43	42	40
83	55	47	44	42	40-41
84	56	48	44	42	40-42
85	56	48	44	43	41-42
86	57	48	45	44	42
87	58	49	46	44	43
88	58	50	46	44	44
89	59	50	47	45	44
90	60	51	48	46	44
91	60	52	48	46	45
92	61	52	48	47	46
93	62	52	48	48	46
94	62	53	49	48	47
95	63	54	50	48	48
96	64	54	50	48	48
97	64	55	51	49	48
98	65	56	52	50	48-49
99	66	56	52	50	48-50
100	66	56	52	50	48-50

Fig. 14 $ur(n,k)$ for $n = 100$ $k \le 6$

$n \backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	1																							
2	2	1																						
3	3	2	1																					
4	4	2	2	1																				
5	5	3	2	2	1																			
6	6	4	3	2	2	1																		
7	7	4	4	3	2	2	1																	
8	8	5	4	4	2	2	2	1																
9	9	6	4	4	3	2	2	2	1															
10	10	6	5	4	4	3	2	2	2	1														
11	11	7	6	5	4	4	3	2	2	2	1													
12	12	8	6	6	4	4	4	3	2	2	2	1												
13	13	8	7	6	5	4	4	4	3	2	2	2	1											
14	14	9	8	7	6	5	4	4	4	3	2	2	2	1										
15	15	10	8	8	7	6	5	4	4	4	3	2	2	2	1									
16	16	10	8	8	8	6	6	5	4	4	4	2	2	2	2	1								
17	17	11	9	8	8	7	6	6	5	4	4	3	2	2	2	2	1							
18	18	12	10	8	8	8	7	6	6	4-5	4	4	3	2	2	2	2	1						
19	19	12	10	9	8	8	8	7	6	5-6	4-5	4	4	3	2	2	2	2	1					
20	20	13	11	10	9	8	8	8	7	6	5-6	4-5	4	4	3	2	2	2	2	1				
21	21	14	12	10	10	8-9	8	8	8	7	6	5-6	4-5	4	4	3	2	2	2	2	1			
22	22	14	12	11	10	8-10	8-9	8	8	8	7	6	4-6	4	4	4	3	2	2	2	2	1		
23	23	15	12	12	11	8-10	8-10	8-9	8	8	8	7	5-6	4-5	4	4	4	3	2	2	2	2	1	
24	24	16	13	12	12	9-11	8-10	8-10	8-9	8	8	8	6	5-6	4	4	4	4	3	2	2	2	2	1

Fig. 15 $w(n,k)$ for $1 \leq k \leq n \leq 24$



The lines k = const. and n-k = const.

Fig. 16

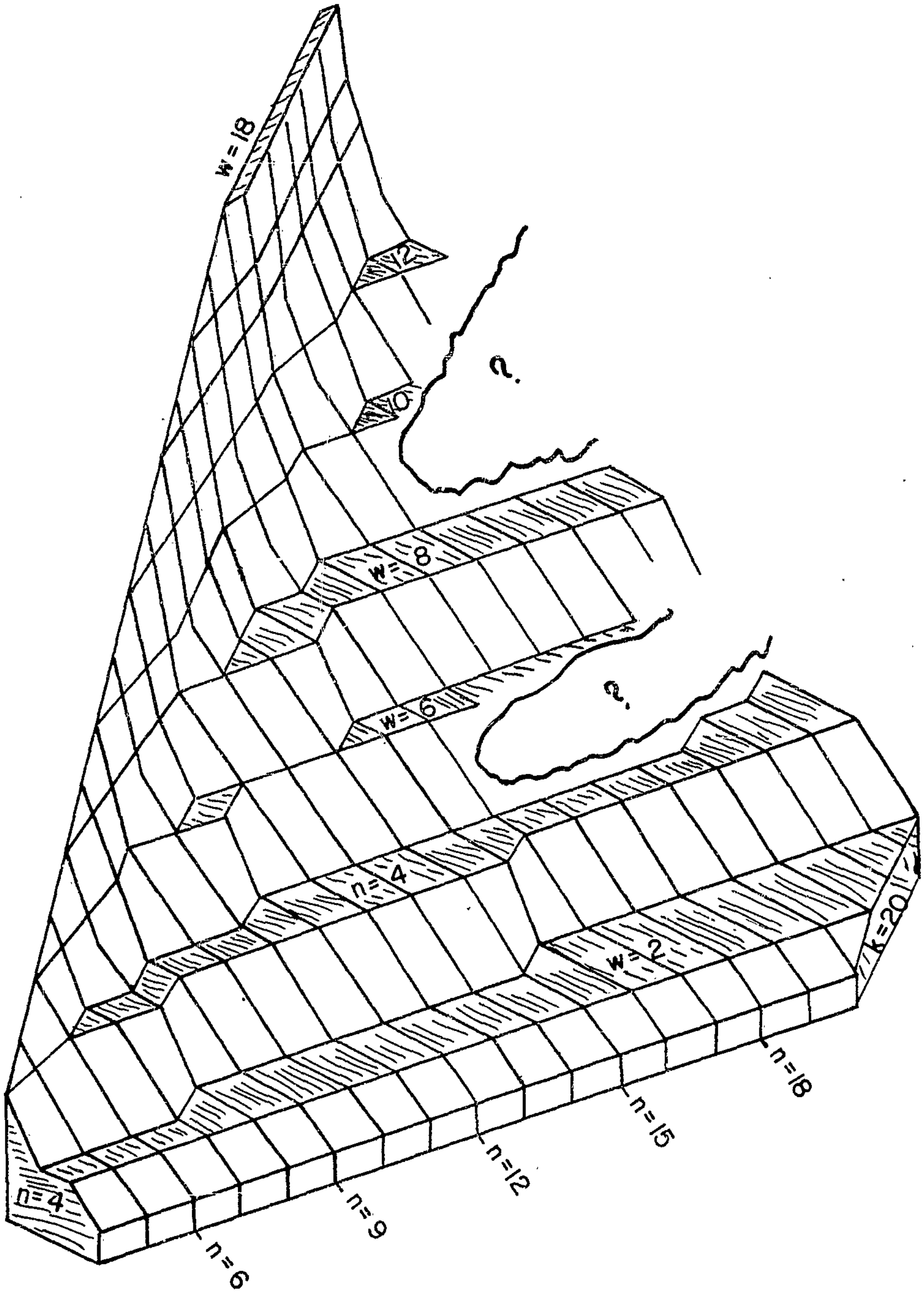


Fig. 17

The lines n = const. and w = even const.

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(Contract AF19(604)-7493)
Unclassified Report

Let w(n,k) be the largest integer such that there exists a binary group code (n,k) all of whose non-zero elements have weight equal to or larger than w(n,k)

In this report values of w(n,k) are given for $0 < k \leq 6$ and $k \leq n \leq 100$, as well as for $0 < k \leq n \leq 24$. Further, new upper and lower bounds are obtained which are easy to compute and, in certain regions, better than other known bounds.

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